1 Coding (Poisson sampling).

Design a Metropolis sampler that targets the Pois(λ) distribution on the nonnegative integers $\{0, 1, 2, \ldots\}$, which has probability mass function $p(x) = e^{-\lambda}\lambda^k/k!$. Implement your sampler with $\lambda = 3, 5, 7$. How well does it work?

2 Coding (Metropolis versus exact sampling).

Implement the exact and single-flip Metropolis samplers for the 1-d Ising model with open boundary conditions. Which sampler works best?

3 Coding (2-d Ising sampling).

Implement the Swendsen-Wang and Wolff samplers for the 2-d Ising model with open boundary conditions. Which sampler works best?

4 Choose-your-own-adventure (Conditional Ising sampling).

Design a Metropolis sampler for the conditional Ising model $p\big(\boldsymbol{x} \mid \sum_i \boldsymbol{x}_i = 0\big)$.

5 Choose-your-own-adventure (Hot and cold temperatures).

Write down the limits of the single-flip Metropolis sampler, the Swendsen-Wang sampler, and the Wolff sampler as $\beta \downarrow 0$ (infinite temperature) or $\beta \uparrow \infty$ (zero temperature). Which samplers are the best and the worst?

6 Math (Potts model).

The Potts model is a generalization of the Ising model in which each spin can take $q \geq 2$ distinct states $x_i \in \{1, 2, \ldots, q\}$. The probability mass function is

$$p(\boldsymbol{x}) = \frac{1}{Z_\beta} \exp\Big(\beta \sum\nolimits_{i\sim j} \delta(x_i - x_j)\Big), \qquad Z_\beta = \sum\nolimits_{\boldsymbol{y}} \exp\Big(\beta \sum\nolimits_{i\sim j} \delta(y_i - y_j)\Big).$$

Write down the generalizations of the single-flip Metropolis, Swendsen-Wang and Wolff samplers for the Potts model. Derive an exact sampler for the 1-d Potts model with open boundary conditions.

7 Math (Converse to Markov chain ergodic theorem).

In class, we showed if x is a positive recurrent state (a state x satisfying $\mathbb{E}[\tau_x \mid X_0 = x] < \infty$ where $\tau_x = \min\{t > 0 : X_t = x\}$ is called positive recurrent), there is a unique stationary measure π (a measure satisfying $\pi(i) = \sum_j \pi(j)p(j,i)$ is called stationary) such that the Markov chain started from $X_0 = x$ satisfies

$$\frac{1}{T}\sum_{t=0}^{T-1} f(X_t) \overset{T\to\infty}{\to} \sum_i \pi(i)f(i)$$

for any bounded function f.

Here, we prove a partial converse. Assume the Markov chain admits a finite stationary measure π. Then, prove that every state x for which $\pi(x) > 0$ is positive recurrent.

(Hint: if μ is a nonnegative vector satisfying $\mu(x) = 1$ and $\mu(i) = \sum_j \mu(j)p(j,i)$ for $i \neq x$, show that $\mu(i) \geq \sum_{t=0}^{\infty} \mathbb{P}\{X_t = i, \tau_x > t \,|\, X_0 = x\}$ for all i.)

8 Math (Communicating classes).

(a) We say that a communicates with b and write $a \leftrightarrow b$ if $\sum_{t=0}^{\infty} p_t(a,b) > 0$ and $\sum_{t=0}^{\infty} p_t(b,a) > 0$. Prove that communication is an equivalence class (it is reflexive, symmetric, and transitive).

(b) For each state i, define the first passage time $\tau_i = \min\{t > 0 : X_t = i\}$. We say that i is *transient* if $\mathbb{P}\{\tau_i < \infty \,|\, X_0 = i\} < 1$ and *recurrent* if $\mathbb{P}\{\tau_i < \infty \,|\, X_0 = i\} = 1$. We say that a recurrent state i is *null recurrent* if $\mathbb{E}[\tau_i \,|\, X_0 = i] = \infty$ and *positive recurrent* if $\mathbb{E}[\tau_i \,|\, X_0 = i] < \infty$. Give examples of transience, null recurrence, and positive recurrence.

(c) Prove that transience, null recurrence, and positive recurrence are class properties, i.e., one of these properties must be true for every state in a communicating class.

(d) Prove that each recurrent communicating class C is absorbing, i.e., $p_t(i, \mathsf{C}) = 0$ for all $t \geq 0$ and $i \in \mathsf{C}$.

(e) Prove that each recurrent communicating class C on a finite state space is positive recurrent.

(f) Prove that there exists a unique stationary measure π for each positive recurrent communicating class C, and it satisfies the formula $\pi(i) = 1/\mathbb{E}[\tau_i \,|\, X_0 = i] > 0$ for each $i \in \mathsf{C}$.