# 1 Coding (Jump up and down and move it all around).

How long does it take for a fractional Brownian motion $(W_t)_{t\geq 0}$ to hit the level $W_t = 1$? Make a histogram of first passage times for $H = 1/4$, $H = 1/2$, and $H = 3/4$ and compare.

# 2 Choose-your-own-adventure (Lemmings problem).

Let's use a CTMC to model lemmings (characters from a 1990s video game) climbing up and falling down a ladder. The system has states $\{0, 1, 2, \ldots, 10\}$, which correspond to the ground level and the ten rungs of the ladder. A lemming at level $i$ climbs up the ladder with a rate 1 for $i = 0, 1, \ldots, 9$. A lemming at level $i$ falls to the ground with a rate 1 for $i = 1, 2, \ldots, 9$. What is the expected time to reach the top starting from the ground? Run a simulation or solve analytically.

**Partial solution.** Set $N = 10$ and $h(i) = \mathbb{E}[\tau_N | X_0 = i]$, where $\tau_N = \min\{t \geq 0 : X_t = N\}$ is the first time to hit $X_t = N$. Then, introduce the rate matrix $\mathbb{Q}$ and use the definition of a CTMC to argue for $1 \leq i \leq N - 1$

$$\begin{aligned} h(i) &= \mathbb{E}[\tau_N | X_0 = i] \\ &= \sum_{j=0}^{N} \mathbb{E}[\tau_N | X_\Delta = j, X_0 = i] \mathbb{P}\{X_\Delta = j | X_0 = j\} \\ &= \sum_{j=0}^{N} \mathbb{E}[\tau_N | X_\Delta = j, X_0 = i] (\delta_{ij} + \Delta \mathbb{Q}_{ij} + \mathcal{O}(\Delta^2)) \end{aligned}$$

We also calculate

$$\begin{cases} \mathbb{E}[\tau_N | X_\Delta = j, X_0 = i] = \Delta + h(j), \quad 1 \leq j \leq N - 1, \\ \mathbb{E}[\tau_N | X_\Delta = N, X_0 = i] \leq \Delta, \end{cases}$$

whence

$$h(i) = h(i) + \Delta + \Delta \sum_{j=0}^{N} \mathbb{Q}_{ij} h(j) + \mathcal{O}(\Delta^2).$$

Dividing by $\Delta$ and taking $\Delta \to 0$, we arrive at

$$0 = 1 + \sum_{j=0}^{N} \mathbb{Q}_{ij} h(j) = 1 + h(0) - 2h(i) + h(i+1)$$

and by rearrangement

$$\begin{aligned} h(i+1) &= 2h(i) - h(0) - 1 \\ &= 4h(i-1) - (2+1)(h(0)+1) \\ &= \cdots \\ &= 2^{i+1} h(0) - (2^i + \cdots + 2 + 1)(h(0) + 1) = h(0) - (2^{i+1} - 1). \end{aligned}$$

Observing $h(N) = 0$, we obtain the solution $h(i) = 2^N - 2^i$ and the expected hitting time is $h(0) = 2^{10} - 1 = 1023$.

## 3 Choose-your-own-adventure (Brownian motion).

What is the probability that a Brownian motion $(W_t)_{t\geq 0}$ hits $W_t = +10$ before $W_t = -1$? What is the expected time for $W_t$ to hit one of the two boundaries? Run a simulation or solve analytically.

## 4 Coding (Michaelis-Menten model).

Let's simulate a CTMC representing four types of molecules in a solvent. We start with $x_1 = 300$ substrate molecules and $x_2 = 100$ enzyme molecules, and we represent the system as a vector $\boldsymbol{X} = (300, 100, 0, 0)^T$. With rate $k_1 = 2 \times 10^{-4} X_1 X_2$, an enzyme binds with a substrate to form an enzyme-substrate complex; $\boldsymbol{X} \leftarrow \boldsymbol{X} + (-1, -1, 1, 0)^T$. With rate $k_2 = 10^{-4} X_2$, an enzyme-substrate complex dissociates back into an enzyme and a substrate; $\boldsymbol{X} \leftarrow \boldsymbol{X} + (1, 1, -1, 0)^T$. With rate $k_3 = 10^{-3} X_3$, an enzyme-substrate complex forms a product (an altered enzyme) and a substrate; $\boldsymbol{X} \leftarrow \boldsymbol{X} + (0, 1, -1, 1)^T$. Plot the number of substrate and product molecules over time.

## 5 Choose-your-own-adventure (Karhunen-Loéve)

(a) One way to approximate a Brownian motion $(W_t)_{0\leq t\leq 1}$ is to linearly interpolate between the time points $t = 0, 1/N, 2/N, \ldots, 1 - 1/N, 1$. What is the mean square error in this approximation? Calculate the error on a computer or analytically.

(b) Another way approximate a Brownian motion is to evaluate the coefficients

$$\langle e_i, W \rangle = \int_0^1 e_i(t) W_t \, dt, \qquad e_i(t) = \sqrt{2} \sin\big((i - 1/2)\pi t\big),$$

and approximate $\hat{W} = \sum_{i=1}^{N} \langle e_i, W \rangle e_i$. What is the mean square error in this approximation?

## 6 Math (Divergence of Euler scheme).

Consider the Euler scheme for solving the SDE $dX = X \, dW$ with initial condition $X_0 = 1$, where $(W_t)_{t\geq 0}$ is a fractional Brownian motion.

(a) Show that the approximation from Euler's scheme

$$\hat{X}_{i\Delta} = \hat{X}_{(i-1)\Delta} + \hat{X}_{(i-1)\Delta}(W_{i\Delta} - W_{(i-1)\Delta}).$$

can be written

$$\hat{X}_T = \exp\Big(\sum_{i=1}^{T/\Delta} \log\big(1 + W_{i\Delta} - W_{(i-1)\Delta}\big)\Big).$$

(b) Applying a Taylor series expansion to part (a), argue that

$$\hat{X}_T = \exp\Big(\sum_{i=1}^{T/\Delta} \Big[W_{i\Delta} - W_{(i-1)\Delta} - \tfrac{1}{2}\big(W_{i\Delta} - W_{(i-1)\Delta}\big)^2 + \cdots\Big]\Big).$$

Identify the limit as $\Delta \to 0$ and the rate of convergence for different values of $H \in (0, 1)$ (hint: use the law of large numbers and the central limit theorem). Note there are other non-Euler schemes that converge faster (Hu, Liu, & Nualart, 2016).

**Partial solution.** We observe that

$$\frac{1}{T/\Delta}\sum_{i=1}^{T/\Delta}\left(\frac{W_{i\Delta}-W_{(i-1)\Delta}}{\Delta^H}\right)^2$$

is a sample average of stationary, decorrelating mean-one random variables, and the weak law of large numbers (e.g., Shao, 1995) guarantees

$$\frac{1}{T/\Delta}\sum_{i=1}^{T/\Delta}\left(\frac{W_{i\Delta}-W_{(i-1)\Delta}}{\Delta^H}\right)^2 = 1+o_p(1) \tag{1}$$

and consequently

$$\sum_{i=1}^{T/\Delta}\left(W_{i\Delta}-W_{(i-1)\Delta}\right)^2 = T\Delta^{2H-1}(1+o_p(1))$$

as $\Delta\to 0$. By examining this expression, we can immediately guarante

$$\begin{cases}\hat{X}\to e^W & \text{at a rate } \Delta^{2H-1} \text{ if } H>1/2,\\ \hat{X}\to 0 & \text{if } H<1/2.\end{cases}$$

The remaining $H=1/2$ case is the subtlest. In this case, we observe the increments $W_{i\Delta}-W_{(i-1)\Delta}$ are independent, and they have second moment $\Delta$ and fourth moment $3\Delta^2$, which implies a central limit theorem

$$\frac{1}{\sqrt{T/\Delta}}\sum_{i=1}^{T/\Delta}\left[\left(\frac{W_{i\Delta}-W_{(i-1)\Delta}}{\Delta^{1/2}}\right)^2-1\right]\overset{\mathcal{D}}{\to}\mathcal{N}(0,2).$$

Consequently, we can sharpen (1) to yield

$$\frac{1}{T/\Delta}\sum_{i=1}^{T/\Delta}\left(\frac{W_{i\Delta}-W_{(i-1)\Delta}}{\Delta^{1/2}}\right)^2 = 1+O_p(\Delta^{1/2})$$

and consequently

$$\sum_{i=1}^{T/\Delta}\left(W_{i\Delta}-W_{(i-1)\Delta}\right)^2 = T+O_p(\Delta^{1/2})$$

as $\Delta\to 0$. We conclude

$$\begin{cases}\hat{X}\to e^W & \text{at a rate } \Delta^{2H-1} \text{ if } H>1/2,\\ \hat{X}\to e^{W-t/2} & \text{at a rate } \Delta^{1/2} \text{ if } H=1/2,\\ \hat{X}\to 0 & \text{if } H<1/2.\end{cases}$$

To make this Taylor series expansion rigorous, make sure to treat the second order term as the remainder term for $H\neq 1/2$ and treat the fourth order term as the remainder term for $H=1/2$. We can talk about how to do this during office hours but warning: it's technical stuff.

# 7 Math (Convergence of Euler scheme).

Consider the Euler scheme for solving the SDE $dX = -X\,dt + dW$ with initial condition $X_0 = 0$, where $(W_t)_{t\geq 0}$ is a fractional Brownian motion.

(a) Show that the approximation from Euler's scheme

$$\hat{X}_{i\Delta} = \hat{X}_{(i-1)\Delta} - \Delta\hat{X}_{(i-1)\Delta} + W_{i\Delta} - W_{(i-1)\Delta}.$$

can be written

$$\hat{X}_T = \sum_{i=1}^{T/\Delta} (1-\Delta)^{T/\Delta - i}\big(W_{i\Delta} - W_{(i-1)\Delta}\big).$$

Turning sums into integrals, write down the limit as $\Delta \to 0$.

(b) Compare the Euler's scheme solution with mesh size $\Delta$ and mesh size $\Delta/2$. Show the difference is a mean-zero Gaussian random variable and bound the variance in terms of $\Delta$ and $H$ (hint: covariances among fractional Gaussian noise increments are negative for $H < 1/2$, positive for $H > 1/2$). What is the convergence rate of Euler's scheme? For more insights, see Butkovsky, Dareiotis, & Gerencsér (2021) and Huang & Wang (2023).

# 8 Math (Making sense of SDEs).

Consider the SDE $dX = b(X)\,dt + \sigma(X)\,dW$, with initial condition $X_0 = x_0$, where $(W_t)_{t\geq 0}$ is a fractional Brownian motion. For appropriate drift functions $b$, diffusion functions $\sigma$, and Hurst parameters $H \in (0,1)$, we can sometimes establish existence and uniqueness using Picard iteration:

$$\hat{X}^{(i)}(t) = x_0 + \int_0^t b(\hat{X}_s^{(i-1)})\,ds + \int_0^t \sigma(\hat{X}_s^{(i-1)})\,dW_s, \qquad \hat{X}^{(0)}(t) = x_0.$$

When does this work? When does this fail? Hint: start with the case of constant $\sigma$ and use Gronwall's inequality.