# Lecture 3: Jump up and down and move it all around

- ☐ Markov chains
- ☐ Continuous-time Markov chains (CTMCs)
- ☐ Fractional Brownian motions (fBMs)
- ☐ Stochastic differential equations (SDEs)

**Ising model** for ferromagnetism has $N$ spins $\vec{x} = (x_0, \ldots, x_{N-1})$ with each $x_i \in \{+1, -1\}$.

Probability mass function

$$p(\vec{x}) = \frac{1}{Z_\beta} \exp\left(\beta \sum_{i=1}^{N-1} x_{i-1} x_i\right), \quad Z_\beta = \sum_{\vec{y} \in \{+1,-1\}^N} \exp\left(\beta \sum_{i=1}^{N-1} y_{i-1} y_i\right)$$

attraction between neighboring spins

$\beta > 0$ inverse temperature parameter

$\beta = 2 \Rightarrow \underbrace{+++++\;----------}_{N=15 \text{ spins}}$ (few transitions)

$\beta = \frac{1}{2} \Rightarrow ++-+--++---+-++$ (many transitions)

**Markov chain representation** we showed

$$p(\vec{x}) = p(x_0)\, p(x_1|x_0)\, p(x_2|x_1) \cdots p(x_{N-1}|x_{N-2})$$

**Markov property**: pmf of $x_t$ only depends on most recent $x_{t-1}$, regardless of $x_{t-2}, x_{t-3}, \ldots, x_0$.

$$x_0 \sim \text{Unif}\{+1,-1\}, \quad x_t = \begin{cases} x_{t-1} & \text{w/ prob } e^{\beta}/(e^{\beta}+e^{-\beta}) \\ -x_{t-1} & \text{w/ prob } e^{-\beta}/(e^{\beta}+e^{-\beta}) \end{cases}$$

<u>Transition matrix</u> We can organize pmf at $t=0$,

$p_0$ into a row vector $p_0^T = \left( \frac{1}{2} \quad \frac{1}{2} \right)$

$\uparrow$ prob that $X_0 = +1$ $\quad$ $\nwarrow$ prob that $X_0 = -1$

and introduce transition matrix $\mathbb{P} = \frac{1}{e^{\beta} + e^{-\beta}} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$

with $\mathbb{P}_{ij} = \mathbb{P}\{ X_t = j \mid X_{t-1} = i \}$

$\Rightarrow$ s-step transition probabilities: $\mathbb{P}\{ X_s = j \mid X_0 = i \} = \underbrace{\mathbb{P}^s(i,j)}_{\text{matrix multiplication}}$

$\Rightarrow$ pmf at time $s$ given by $p_s^T = p_0^T \mathbb{P}^s$.

<u>Simulation of Markov chains</u>

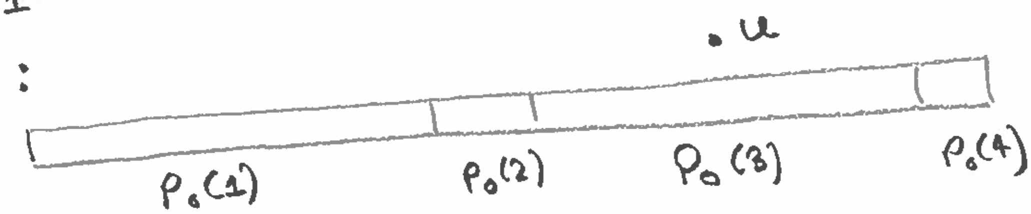
Input: starting distribution $p_0$, transition matrix $\mathbb{P}$, time $T$

Output: Sample $X_0, X_1, \dots, X_T$

1. Simulate $U_0 \sim \text{Unif}(0,1)$
2. Set $X_0 = i$, where $i$ is smallest integer s.t.

$$\sum_{j=1}^{i} p_0(j) = p_0(1) + \dots + p_0(i) > u.$$

Ex:




3. For $t = 1, 2, \dots, T$:
   a) Simulate $U_t \sim \text{Unif}(0,1)$.
   b) Set $X_t = i$, where $i$ is smallest integer s.t.
   $\sum_{j=1}^{i} \mathbb{P}_{X_{t-1} j} = \mathbb{P}_{X_{t-1} 1} + \dots + \mathbb{P}_{X_{t-1} i} > u_t$.

Ex Ising model w/ $\beta = \frac{\log 2}{2} \Rightarrow \mathbb{P} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$

$(U_0, U_1, \ldots, U_4) = (.58, .22, .85, .29, .18)$

$\vec{X} = \underbrace{- + - + +}_{\text{many transitions}}$

Ising model w/ $\beta = \log 2 \Rightarrow \mathbb{P} = \begin{pmatrix} 4/5 & 1/5 \\ 1/5 & 4/5 \end{pmatrix}$

$(U_0, U_1, \ldots, U_4) = (.58, .22, .85, .29, .18)$

$\vec{X} = \underbrace{- - - - +}_{\text{few transitions}}$

Eigendecomposition $\mathbb{P} = 1 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

Correlation function for Ising model:

$$C(t) = \mathbb{E}[X_0 X_t] = \mathbb{P}\{X_0 = X_t\} - \mathbb{P}\{X_0 = -X_t\} = 2\mathbb{P}\{X_0 = X_t\} - 1$$

$$= 2 \sum_i p_0(i) \underbrace{\mathbb{P}^t(i,i)}_{\substack{t\text{-step transition probabilities} \\ \text{from matrix multiplication}}} - 1 = \sum_i \mathbb{P}^t(i,i) - 1 = \operatorname{tr}(\mathbb{P}^t) - 1$$

$$= \lambda_1(\mathbb{P}^t) + \lambda_2(\mathbb{P}^t) - 1 = \lambda_1(\mathbb{P})^t + \lambda_2(\mathbb{P})^t - 1 = \left( \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} \right)^t.$$

$\Rightarrow$ Since $\left( \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} \right)^t = \left( 1 - \frac{2}{e^{2\beta} + 1} \right)^t$, we have exponentially decreasing correlations w/ lengthscale $\ell \approx \frac{e^{2\beta} + 1}{2}$.

# Continuous-time Markov chains (CTMCs)

- Instead of $t = 0, 1, 2, \dots$, consider all times $t \geq 0$.
- Instead of transition matrix $P \in \mathbb{R}^{N \times N}$ with

$$P_{ij} \geq 0 \quad \forall \; 1 \leq i, j \leq N \quad \text{and} \quad \sum_{j=1}^{N} P_{ij} = 1 \quad \text{for} \quad 1 \leq i \leq N,$$

consider <u>rate</u> matrix $Q \in \mathbb{R}^{N \times N}$ with

$$Q_{ij} \geq 0 \quad \forall \; i \neq j \quad \text{and} \quad \sum_{j=1}^{N} Q_{ij} = 0 \quad \text{for} \quad 1 \leq i \leq N.$$

- Instead of $p_{s+1}^T = p_s^T P \Rightarrow \boxed{p_{s+1}^T - p_s^T = p_s^T (P - I)}$

we have $p_{s+\Delta}^T = p_s^T + \Delta p_s^T Q + O(\Delta^2) \Rightarrow \boxed{\frac{d}{ds} p_s^T = p_s^T Q}$

- Process can jump in any time interval $(s, s+\Delta)$

## Simulation of CTMCs

Input: starting distribution $p_0$, rate matrix $Q$, time $T > 0$.

Output: Sample $(X_t)_{0 \leq t \leq T}$.

1. Simulate $u \sim \text{Unif}(0,1)$.
2. Set $y_0 = i$, where $i$ is smallest integer s.t.
   $\sum_{j=1}^{i} p_0(j) > u$.
3. Set $k = 0$, $t_0 = 0$.
4. while $t_k < T$:
   a) Set $k = k + 1$.
   b) Simulate $u_k \sim \text{Unif}(0,1)$.
   c) Set $t_k = t_{k-1} + \log u_k \,/\, Q_{y_{k-1}, y_{k-1}}$.

d) Simulate $V_K \sim \text{Unif}(0,1)$.

e) Set $y_K = i$, where $i$ is smallest integer s.t.

$$\sum_{\substack{j=1 \\ j \neq y_{K-1}}}^{i} Q_{y_{K-1}, j} > V_K \, |Q_{y_{K-1}, y_{K-1}}|.$$

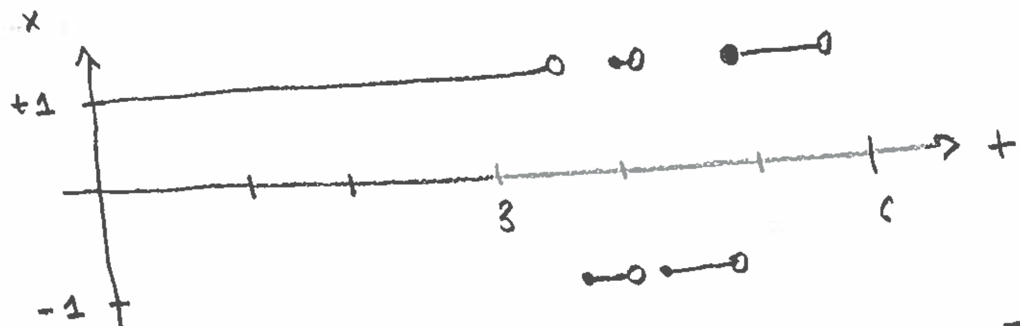
f) Set $X_t = y_{K-1}$ for $t_{K-1} \leq t < t_K$.

<u>Ex</u> Continuous-time Ising model

$$p_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \qquad Q = \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix} \quad \text{for } \alpha > 0.$$

With $\alpha = 1$, I simulated $(u_0, t_1, t_2, \ldots, t_5)$

$= (.27, 3.6, 4.0, 4.2, 4.8, 5.7)$




<u>Time evolution</u> $\frac{d}{ds} p_s^T = p_s^T Q \Rightarrow p_s^T = p_0^T \exp(sQ), \; s > 0$

<u>Ex</u> Ising model $Q = 0 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} (1/\sqrt{2} \;\; 1/\sqrt{2}) + (-2\alpha) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} (1/\sqrt{2} \;\; -1/\sqrt{2})$

$$\Rightarrow \exp(sQ) = \underset{\uparrow}{P^s} = 1 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} (1/\sqrt{2} \;\; 1/\sqrt{2}) + e^{-2\alpha s} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} (1/\sqrt{2} \;\; -1/\sqrt{2})$$

transition probabilities over time $s$

$\Rightarrow$ Aligns with discrete-time Ising model for

$$e^{-2\alpha} = \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}}$$

$\Rightarrow$ Ising model is discrete snapshot of two-state CTMC with rate

$$\alpha = -\frac{1}{2}\log\left(\frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}\right) \approx \frac{1}{e^{2\beta}+1}.$$

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## Fractional Brownian motion (fBM)

Q: What if we want continuous paths?

- ☒ Poisson point process
- ☒ CTMC
- ☑ Gaussian process

Recall $(X_t)_{t\geq 0}$ is mean-zero Gaussian process if $(X_{t_1}, X_{t_2}, \dots, X_{t_k}) \sim \mathcal{N}(0, \Sigma)$ for each $(t_1, t_2, \dots, t_k)$.

Here, $\Sigma = (\Sigma_{ij})_{i,j \in \{t_1,\dots,t_k\}}$ is positive semidefinite matrix w/ $\Sigma_{ts} = \mathbb{E}[X_t X_s] = \frac{1}{2}\Big(\mathbb{E}X_t^2 + \mathbb{E}X_s^2 - \underbrace{\mathbb{E}|X_t - X_s|^2}_{\text{increments}}\Big)$

fBM is mean-zero Gaussian process with $W_0 = 0$ and increments $\mathbb{E}|W_t - W_s|^2 = |t-s|^{2H}$ for a "Hurst parameter" $0 < H < 1$

Scaling $X_t = \frac{1}{\Delta^H} W_{\Delta t}$ has increments $\mathbb{E}|X_t - X_s|^2$

$= \frac{1}{\Delta^{2H}} \mathbb{E}|W_{\Delta t} - W_{\Delta s}| = |t-s|^{2H}$ and is fBM for $\Delta > 0$.

$\Rightarrow$ To sample fBM $(W_0, W_\Delta, W_{2\Delta}, \dots, W_{T\Delta})$, just sample fBM $(X_0, X_1, \dots, X_T)$ and multiply by $\Delta^H$.

<u>Q</u> How do we sample $(W_0, W_1, \dots, W_T)$?

<u>A</u> Since $W_0 = 0$ it suffices to sample <u>fractional Gaussian noise</u> (fGn) $G_t = W_t - W_{t-1}$ for $1 \le t \le T$.

1. $(G_t)_{1 \le t \le T}$ is mean-zero Gaussian process with covariance

$$\Sigma_{t,t+h} = C(h) = \mathbb{E}\left[(W_1 - W_0)(W_{h+1} - W_h)\right]$$

$$= \mathbb{E}\left[W_1(W_{h+1} - W_h)\right] = \frac{1}{2}\left(|h+1|^{2H} - 2h^{2H} + |h-1|^{2H}\right)$$

2. It suffices to simulate from a $\mathcal{N}(0,\Sigma)$ distribution where

$$\Sigma = \begin{pmatrix} c(0) & c(1) & \cdots & c(T-1) \\ c(1) & c(0) & & \\ \vdots & & \ddots & \\ c(T-1) & & & c(0) \end{pmatrix}$$

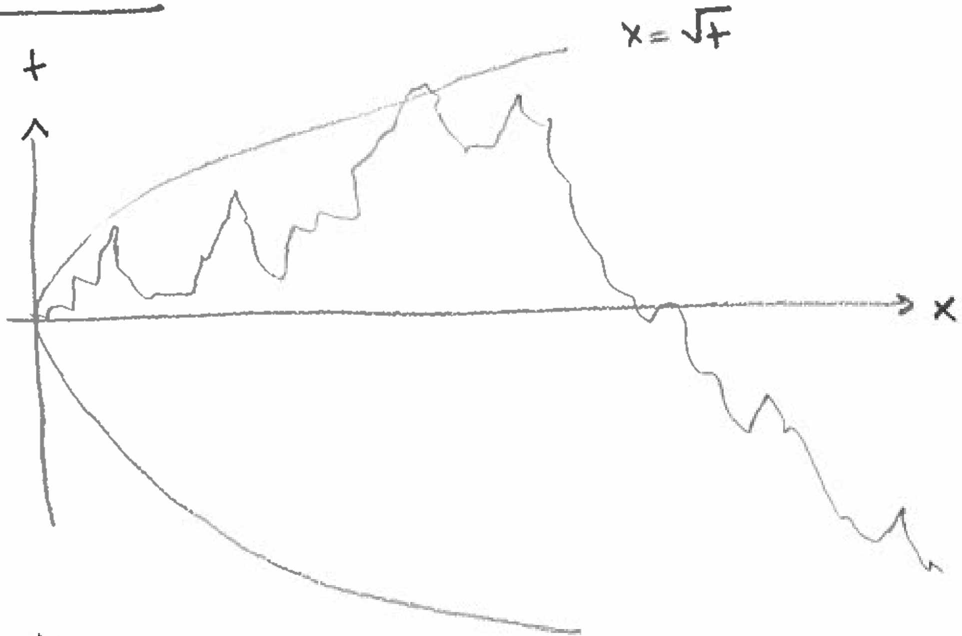
3. For example, we can compute the Cholesky distribution $\Sigma = CC^T$, simulate independent Gaussians $\vec{z} = (z_1, \dots, z_T)$ and set $\vec{G} = C\vec{z}$. There is also a faster $O(T \log T)$ method using circulant embedding (Perrin, Harba, Jennane, & Iribarren, 2002)

<u>BM</u> is even simpler to sample since $H = \frac{1}{2}$

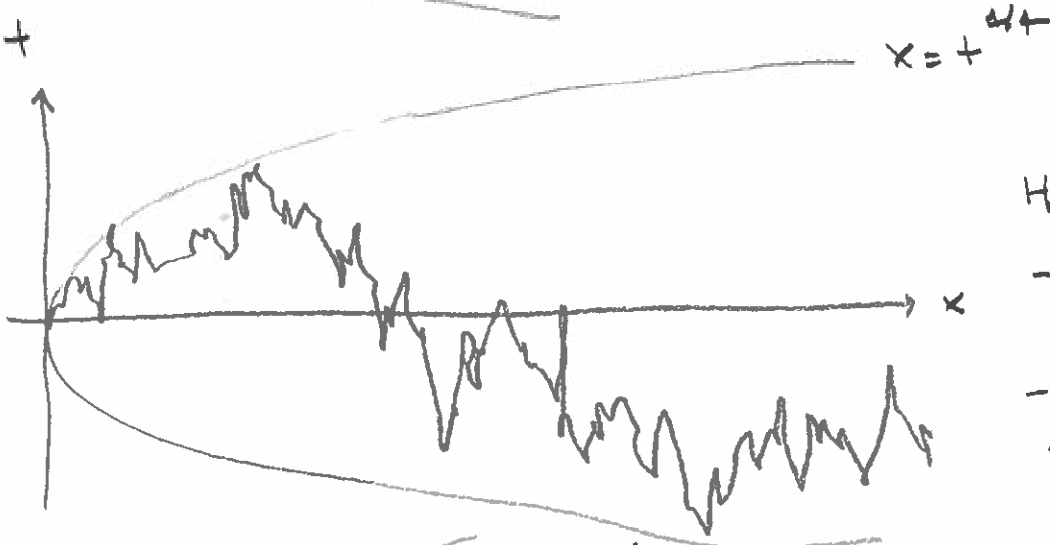
$\Rightarrow c(h) = \begin{cases} 1, & h = 0 \\ 0, & \text{otherwise} \end{cases}$ so the Gaussian noise is a sequence of independent Gaussians.

Q Now that we can simulate fBM, what can we do?

# Plot it

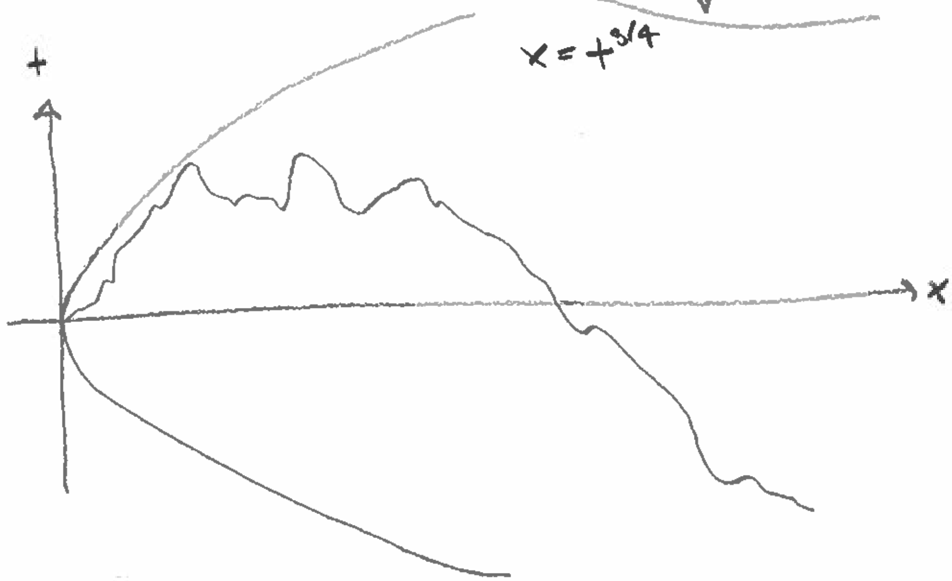



$H = \frac{1}{2}$ Brownian motion




$H = \frac{1}{4}$ subdiffusion

- jumpy on short time frames
- constrained on long time frames




$H = \frac{3}{4}$ superdiffusion

- smoother on short time frames
- travels over long time frames

## Stochastic differential equations (SDEs)

- Simplest one-dimensional case is

$$dX = \underbrace{b(X)\,dt}_{\text{"drift"}} + \underbrace{\sigma(X)\,dW}_{\text{"diffusion" / "anomalous diffusion"}}$$

- This is like an ODE with random forcing

$$\frac{dX}{dt} = \underbrace{b(x)}_{\text{normal part}} + \underbrace{\sigma(X)\frac{dW}{dt}}_{\text{weird part}}$$

## Simulation of SDEs:

For "nice" SDEs, the Euler discretization works well enough.

Input: Initial condition $x_0$, drift $b$, diffusion $\sigma$, time $T \geq 0$, mesh size $\Delta > 0$, Hurst parameter $H \in (0,1)$

Output: Time series $(X_{i\Delta})_{0 \leq i \leq T/\Delta}$ of approximate sample

1. Set $X_0 = x_0$.
2. Generate fGn $(G_i)_{1 \leq i \leq T/\Delta}$.
3. For $i = 1, 2, \ldots, T/\Delta$:

   Set $X_{i\Delta} = X_{(i-1)\Delta} + \Delta\, b(X_{(i-1)\Delta}) + \Delta^H \sigma(X_{(i-1)\Delta}) G_{i\Delta}$.

## Convergence

- Euler discretization converges to the SDE solution (in "strong" sense) with rate

$$\begin{cases} \Delta, & H \geq \frac{1}{2} \\ \Delta^{2H}, & H < \frac{1}{2} \end{cases}$$

Kind of slow...

if $\sigma$ is constant, $b$ is Lipschitz.

- Euler discretization converges with slower rate

$$\begin{cases} \Delta^{2H-1}, & H > \frac{1}{2} \\ \Delta^{1/2}, & H = \frac{1}{2} \end{cases}$$

and diverges for $H < \frac{1}{2}$ if $\sigma$ is nonconstant, e.g. $dX = X\,dW$ $(\sigma(x) = x)$.

<u>Caution</u> Rules of calculus fail for $H = \frac{1}{2}$, need new rules called Itô calculus.

<u>Ex: geometric Brownian motion</u> $dX = X\,dW$

has solution
$$\begin{cases} X_t = x_0 \exp(W_t), & H > \frac{1}{2} \\ X_t = X_0 \exp\left(W_t - \frac{1}{2}t^2\right), & H = \frac{1}{2} \end{cases}$$

↖ should be $\exp(W_t - t/2)$

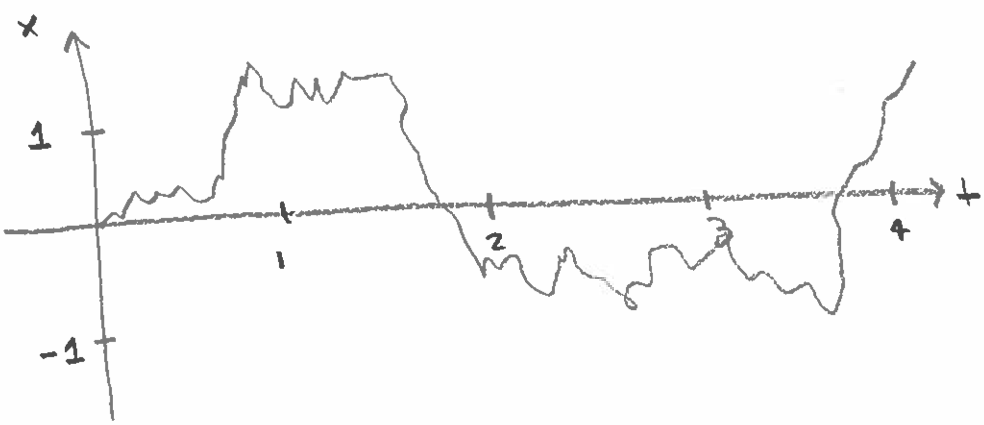
Spooky

<u>Ex: Ornstein-Uhlenbeck process</u> $dX = -X\,dt + dW$

$\Rightarrow \sigma(x) = 1$ is constant so we're okay to integrate.

$$\frac{dX}{dt} = -X + \frac{dW}{dt} \Rightarrow \frac{d}{dt}\left(e^t X\right) = e^t\left(X + \frac{dX}{dt}\right) = e^t \frac{dW}{dt}$$

$$\Rightarrow e^t X_t - x_0 = \int_0^t e^s\,dW \Rightarrow \boxed{X_t = e^{-t} x_0 + \int_0^t e^{s-t}\,dW}$$




$H = \frac{1}{2}$