Agenda

- Definition of point processes in $\mathbb{R}^d$
- How to describe/characterize the distribution of a PP
- Poisson processes (Defn & Simulation)
- Stochastic Geometry Models built from Poisson processes.
- Extra: Non-Poisson point processes

$\boxed{\text{Point processes in } \mathbb{R}^d}$

Def A subset $\varphi \subseteq \mathbb{R}^d$ is locally finite if for all bounded $B \in \mathcal{B} := \mathcal{B}(\mathbb{R}^d)$, $\varphi(B) := \#(\varphi \cap B)$ is finite.

$N := \{ \varphi \subseteq \mathbb{R}^d : \varphi \text{ is locally finite} \}$

$\mathcal{N} :=$ σ-algebra generated by subsets of the form:
$\{ \varphi \in N : \varphi(B) = k \}$ for bounded $B \in \mathcal{B}$ and $k \in \mathbb{N} \cup \{0\}$.

Def A point process Φ is a random variable taking values in $(N, \mathcal{N})$

Ex 1: Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be iid random vectors in $\mathbb{R}^d$ with probability density f.

$\Phi = \{X_i\}_{i=1}^n$ is a Binomial point process.

$\to \Phi(B) \sim \text{Binomial}(n, \int_B f(x)dx)$

In general, we can write any point process Φ on $\mathbb{R}^d$ as $\Phi = \{X_i\}_{i=1}^N$ for random vectors $X_i \in \mathbb{R}^d$ and a random variable $N \in \mathbb{N} \cup \{\infty\}$.

Intensity Measure

Def The intensity measure of a point process Φ on $\mathbb{R}^d$ is the measure Λ on $(\mathbb{R}^d, \mathcal{B})$ defined by:

$$\Lambda(B) := \mathbb{E}[\Phi(B)] \quad \text{for } B \in \mathcal{B}.$$

$$= \text{Expected number of points of } \Phi \text{ in } B$$

Note: Even though Φ is locally finite almost surely, for bounded $B \in \mathcal{B}$, $\mathbb{P}[\Phi(B) < \infty] = 1 \not\Rightarrow \mathbb{E}[\Phi(B)] < \infty$

We will later assume $\Lambda(B) < \infty$ $\forall$ bounded $B \in \mathcal{B}$.

Def If Λ has a density λ w.r.t. Lebesgue measure, (i.e. $\Lambda(B) = \int_B \lambda(x)\,dx$), then we say Φ has intensity function $\lambda(\cdot)$.

Def If Λ has a constant density $\lambda > 0$ w.r.t. Lebesgue measure (i.e. $\Lambda(B) = \lambda \operatorname{vol}(B)$), then Φ is homogenous with intensity λ.

Excercise: What is the intensity measure of the Binomial point process in example 1?

Def The finite dimensional (fidi) distributions of a point process Φ are the distributions of the random vectors $(\Phi(B_1), \dots, \Phi(B_n))$ for $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{B}$.

Prop 1 Let Φ and Φ' be point processes. If for all $n \in \mathbb{N}$ and pairwise disjont $B_1, \dots, B_n \in \mathcal{B}$,

$$(\Phi(B_1), \dots, \Phi(B_n)) \stackrel{d}{=} (\Phi'(B_1), \dots, \Phi'(B_n)),$$

then $\Phi \stackrel{d}{=} \Phi'$. That is, fidi distributions characterize the distribution of a point process.

Poisson Processes

Let Λ be a locally finite measure on $\mathbb{R}^d$, i.e. $\Lambda(B) < \infty$ for all bounded $B \in \mathcal{B}$.

<u>Def</u> A point process Φ on $\mathbb{R}^d$ is a Poisson process with intensity measure Λ if:

(i) For all $B \in \mathcal{B}$ s.t. $\Lambda(B) < \infty$, $\Phi(B) \sim \text{Poisson}(\Lambda(B))$

(ii) For all $n \in \mathbb{N}$ and pairwise disjoint $B_1, \dots, B_n \in \mathcal{B}$, $\Phi(B_1), \dots, \Phi(B_n)$ are independent.

That is,

$$\mathbb{P}[\Phi(B_1) = k_1, \dots, \Phi(B_n) = k_n] = \prod_{i=1}^{n} \frac{\Lambda(B_i)^{k_i}}{k_i!} e^{-\Lambda(B_i)}.$$

<u>Ex2</u>: Let $N \sim \text{Poisson}(\gamma)$ for $\gamma > 0$. Let $X_1, \dots, X_N$ be iid random vectors in $\mathbb{R}^d$ with probability density f. Then, $\Phi = \{X_i\}_{i=1}^{N}$ is a poisson process with intensity function γf.

<u>Exercise</u>: Prove the claim in example 2 (Hint: use Prop 1)

<u>Simulation</u>

Let $E \subseteq \mathbb{R}^d$ be such that $\Lambda(E) < \infty$.

To simulate the points of a point process Φ with intensity measure Λ on E:

1. Sample $N \sim \text{Poisson}(\Lambda(E))$.
2. Conditioned on $N = n$, sample $X_1, \dots, X_n$ iid from distribution $\dfrac{\Lambda(\cdot \cap E)}{\Lambda(E)}$

- If Φ is homogenous with intensity $\lambda > 0$

  $\Lambda(E) = \lambda \operatorname{vol}(E)$, $\dfrac{\Lambda(\cdot \cap E)}{\Lambda(E)} \sim \text{Uniform}(E)$

- If Φ has intensity function λ with $\lambda(x) = \lambda^* \ \forall x \in E$:

  ① Sample a homogenous Poisson process Φ^* with intensity λ^* on E.

  ② Let $p(x) = \dfrac{\lambda(x)}{\lambda^*}$. Conditioned on $\Phi^* = \{x_i\}_{i=1}^{n}$, for each i, independently delete x_i with probability $p(x_i)$.

* In general, an independent thinning as above of any Poisson process is again a Poisson process (see exercise on last page)

Stochastic Geometry Models Built from Poisson Processes

Idea: Use transformations that preserve "Poisson-ness"

① Marking: Let $(M, \mathcal{M})$ be a CSMS (complete, separable, metric space) called the mark space

$\Phi = \{X_i\}_{i=1}^{N}$ a Poisson process on $\mathbb{R}^d$ w/ intensity measure Λ

$\{Y_i\}_{i=1}^{N}$ a collection of random elements of $(M, \mathcal{M})$ with distribution $\mathbb{Q}$ independent of Φ

Thm $\hat{\Phi} = \{(X_i, Y_i)\}_{i=1}^{N}$ is a Poisson process on $\mathbb{R}^d \times M$ with intensity measure $\hat{\Lambda}(dx \times dy) = \Lambda(dx)\,\mathbb{Q}(dy)$

② <u>Mapping</u>: Let $(E, \mathcal{E})$ be a CSMS.

Let $T : (\mathbb{R}^d, \mathcal{B}) \to (E, \mathcal{E})$ be a measurable mapping.

$\Phi = \{X_i\}_{i=1}^N$ a Poisson process on $\mathbb{R}^d$ w/ intensity measure Λ

<u>Thm</u> $\tilde{\Phi} = T(\Phi) = \{T(X_i)\}_{i=1}^N$ is a Poisson process on E with intensity measure $T(\Lambda) = \Lambda \circ T^{-1}$

<u>Ex3</u>: Stationary and isotropic Poisson hyperplane process.

Let $\Phi = \{t_i\}_{i=1}^\infty$ be a homogenous Poisson process on $\mathbb{R}$ with intensity λ

Let $\{u_i\}_{i=1}^\infty$ be a collection of iid random vectors uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ independent of Φ

$\Rightarrow \hat{\Phi} = \{(t_i, u_i)\}_{i=1}^\infty$ is a Poisson process on $\mathbb{R} \times \mathbb{S}^{d-1}$ with intensity measure $\Lambda(dt \times du) = \lambda\, dt\, \underset{\text{uniform measure}}{\sigma(du)}$

Consider the measurable mapping

$$T : (t, u) \to H(u,t) := \{x \in \mathbb{R}^d : \langle x, u \rangle = t\}.$$

Then, $T(\hat{\Phi}) = \{H(u_i, t_i)\}_{i=1}^\infty$ is a Poisson process in the space $\mathcal{H}^d$ of hyperplanes in $\mathbb{R}^d$ with intensity measure $\tilde{\Lambda}(\cdot) = \lambda \int_{\mathbb{R}} \int_{\mathbb{S}^{d-1}} 1_{\{H(u,t) \in \cdot\}}\, d\sigma(u)$

- $\mathcal{H}^d$ is equipped with the hit-miss topology.
- For compact sets C in $\mathbb{R}^d$, sets of the form:

$$[C] := \{H \in \mathcal{H}^d : H \cap C \neq \emptyset\}$$

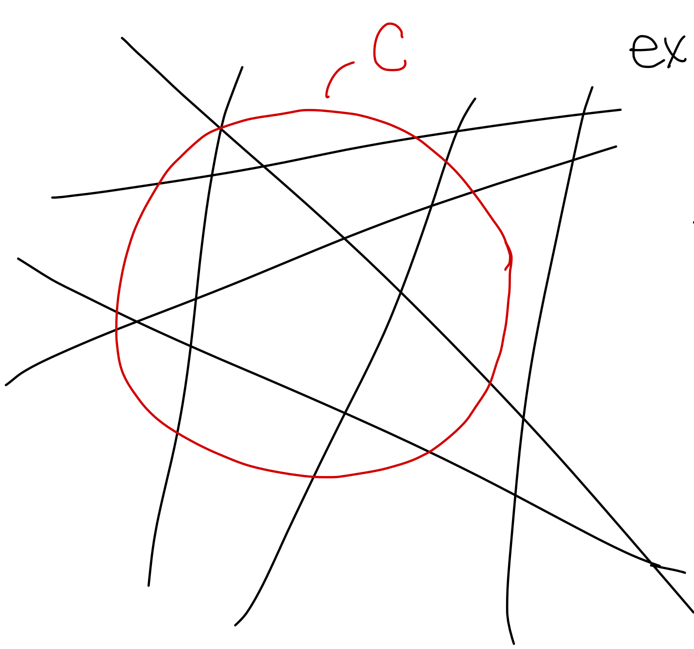
are compact sets of $\mathcal{H}^d$.

- $\tilde{\Phi} := T(\hat{\Phi}) = \{H(u_i, t_i)\}_{i=1}^{\infty}$ is a Poisson process on $\mathcal{H}^d$ with intensity measure $\tilde{\Lambda}$ $\Rightarrow$

$\tilde{\Phi}([C])$ = number of hyperplanes in $\tilde{\Phi}$ that have non-empty intersection with C

$$\sim \text{Poisson}(\tilde{\Lambda}([C])),$$

- $\tilde{\Lambda}([C]) = \lambda \int_{\mathbb{R}} \int_{S^{d-1}} \mathbb{1}\{H(u,t) \cap C \neq \emptyset\} \, d\sigma(u) \, dt$



ex: If C is $B(0,R)$ a ball of radius $R > 0$ centered at 0,

$$\tilde{\Lambda}([B(R)]) = \lambda \int_{\mathbb{R}} \mathbb{1}\{|t| \leq R\} \, dt = 2R\lambda$$

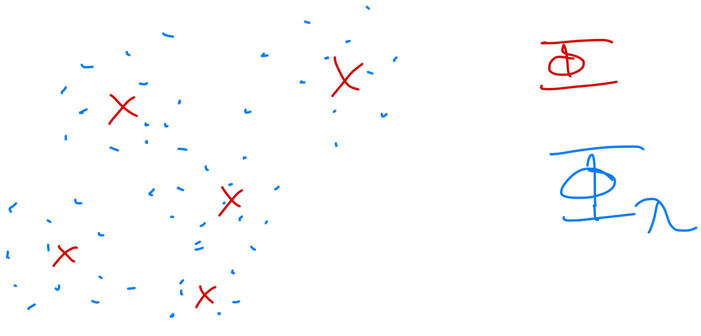
Non-Poisson Point processes

Cox Processes: Poisson processes with a random intensity measure

ex: Let $\Phi = \{X_i\}_{i=1}^{N}$ be a Poisson process on $\mathbb{R}^d$.
Define the random intensity function

$$\lambda(x) := \gamma \sum_{X_i \in \Phi} p(x - X_i), \quad x \in \mathbb{R}^d$$

where p is a probability density on $\mathbb{R}^d$.
Let Φ_λ be the point process such that conditioned on λ, Φ_λ is a Poisson process with intensity function λ. Then, Φ_λ is called a Poisson cluster process.



If $p(x) \propto 1\{\|x\| \leq R\}$, the Φ_λ is called a Mátern cluster process

- Cox processes are "attractive"
- "Repulsive" point processes
  - → Some Gibb point processes (often can only approximately sample)
  - → Determinantal point process (exact sampling algorithm)