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Lecture 1: How random number generators (don't) work

ACM 206, April 4, 2023

Samples inside a circle	Higher dimensions	A better algorithm
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Samples inside a circle		

Let's generate samples inside a circle.



Figure: A Monte Carlo approach (10 samples)

Samples inside a circle	Higher dimensions	A better algorithm
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Samples inside a circle		

Let's generate samples inside a circle.



Figure: A Monte Carlo approach (100 samples)

Let's generate samples inside a circle.



Figure: A Monte Carlo approach (250 samples)

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Let's generate samples inside a circle.

Fraction 0.774

Figure: A Monte Carlo approach (1000 samples)

Higher dimensions

A simple rejection sampling approach

A simple rejection sampling approach

1. For $t = 1, 2, \ldots, T$, draw a random vector $X^t = (X_1^t, X_2^t)$ where

 $X_1^t, X_2^t \sim \mathsf{Unif}(-1, 1).$

2. Accept if

 $(X_1^t)^2 + (X_2^t)^2 \le 1.$

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Figure: A Monte Carlo approach (10 samples)

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Figure: A Monte Carlo approach (100 samples)





Figure: A Monte Carlo approach (250 samples)



Fraction 0.002

Figure: A Monte Carlo approach (1000 samples)

	Higher dimensions	A better algorithm
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Higher dimensions		

Here's a better algorithm for higher dimensions.



Samples inside a circle	Higher dimensions	A better algorithm
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Higher dimensions		

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Here's a better algorithm for higher dimensions.

Don't throw it like this.



Figure: Unif(-1, 1) distribution



Here's a better algorithm for higher dimensions.

1.50 1.50 1.50 Lopapility density *b*(*x*) 1.25 1.00 0.75 0.50 0.25 1.50 1.25 1.00 0.75 0.50 0.25 0.75 0.50 0.00 0.00 <u>↓</u> _1.5 -1.0 -1.01.0 -0.50.0 0.5 1.0 1.5 -0.50.0 0.5 1.5 Value x Value x

Figure: Unif(-1, 1) distribution

Don't throw it like this.

Figure: N(0, 1/(d-1)) distribution

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Throw it like this instead.

Samples inside a circle	Higher dimensions	A better algorithm
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Higher dimensions		

Here's a better algorithm for higher dimensions.



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Higher dimensions

Here's a better algorithm for higher dimensions.

A faster Monte Carlo approach

1. For $t = 1, 2, \dots, T$, draw a random vector $X^t = (X_1^t, \dots, X_d^t)$, where

$$X_1^t,\ldots,X_d^t\sim\mathcal{N}(0,rac{1}{d-1}).$$

2. Accept with probability

$$\exp\left(\frac{d-1}{2}((X_1^t)^2 + \dots + (X_d^t)^2 - 1)\right)\mathbb{1}\left\{(X_1^t)^2 + \dots + (X_d^t)^2 \le 1\right\}$$

Samples inside a circle	Higher dimensions	A better algorithm
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10-dimensional hyperspher	re	



Figure: A faster Monte Carlo approach (10 samples)

Samples inside a circle 00	Higher dimensions O	A better algorithm
10-dimensional hyperspher	re	



Figure: A faster Monte Carlo approach (100 samples)

Samples inside a circle	Higher dimensions	A better algorithm
00	O	○○●○
10-dimensional l	hypersphere	



Figure: A faster Monte Carlo approach (250 samples)





Figure: A faster Monte Carlo approach (1000 samples)

Samples inside a circle	Higher dimensions	A better algorithm
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Conclusion

Why is the Gaussian sampler so effective?



Samples inside a circle	Higher dimensions	A better algorithm
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Conclusion		

Why is the Gaussian sampler so effective?



Figure: Each coordinate is quite close to a $N\left(0,\frac{1}{d-1}\right)$ distribution.

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	Higher dimensions	A better algorithm
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Conclusion		
Conclusion		

Why is the Gaussian sampler so effective?



Figure: Each coordinate is quite close to a $N\left(0, \frac{1}{d-1}\right)$ distribution.

Takeaway message: to speed up Monte Carlo, use everything you know about the events you're sampling!



Why is the Gaussian sampler so effective?



Figure: Each coordinate is quite close to a $N\left(0, \frac{1}{d-1}\right)$ distribution.

Takeaway message: to speed up Monte Carlo, use everything you know about the events you're sampling!

Faster option: There is also a perfect sampler (Barthe et al., 2005):

$$(Z_1,\ldots,Z_n)/(\sum Z_i^2+S)^{1/2}$$

where Z_i are independent Gaussians, $S \sim \exp(1/2)_{i}$, $z \to z \to z$, $z \to z \to z$